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Automatic Control Laboratory

Subspace Identification of Linear Time-Periodic Systems with Periodic Inputs

Mingzhou Yin, Dr. Andrea Iannelli, Prof. Roy S. Smith

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Periodic systems, but why?

- **Linear time-periodic (LTP) systems**: systems with periodically varying linear dynamics
- Periodicity comes naturally in practical systems
  - Rotating dynamics (Allen, Sracic, et al. 2011)
  - Periodic scheduling parameters (Felici, Wingerden, and Verhaegen 2007)
  - Periodic operating trajectory (Allen and Sracic 2009)
- Linear time-periodic (LTP) systems as an intermediate step for
  - LTV systems
  - LPV systems
  - nonlinear systems along limit cycles

The problem

LTP identification problem

Consider a discrete-time stable LTP system with minimal state-space model

$$\begin{cases} x(t+1) &= A_t x(t) + B_t u(t) \\ y(t) &= C_t x(t) \end{cases}, A_t = A_{t+P}, B_t = B_{t+P}, C_t = C_{t+P}, \tag{1}$$

P is the known period.

Problem: Estimate a state-space model equivalent to (1) from input-output measurements with output noise.

An gap of frequency domain methods

- The prevailing method: **time-domain subspace identification**
  - (Verhaegen and Yu 1995), (Hench 1995)

- However, study of frequency domain methods is limited...
- ...**why needed?** To make use of periodic data, for
  - avoiding initial state estimation
  - easier time-domain averaging
- ...**why difficult?** Frequency domain behavior is different from LTI systems

Frequency response of LTP systems

- Response at each frequency ω is not independent
- An input at ω will generate response at $\omega + 2k\pi/P,\ k \in \mathbb{Z}$
- Complex gain $G(\omega)$ becomes a function $G_\omega(\omega + 2k\pi/P)$ of k.

Two paths to address this problem:

- Identify $G_\omega(\omega + 2k\pi/P)$ directly at each frequency $\rightarrow$ frequency lifting
  - (Uyanik et al. 2019)
  - Very restrictive in input design to avoid overlap of frequency response contents
- **In this work,** make use of LTI reformulation

On the shoulders of LTI systems

- LTP systems as structured LTI systems

Time-lifting

Concatenate inputs and outputs of one period

$$\tilde{u}(k) = \begin{bmatrix} u^\top(kP) & u^\top(kP+1) & \cdots & u^\top(kP+P-1) \end{bmatrix}^\top,$$
$$\tilde{y}(k) = \begin{bmatrix} y^\top(kP) & y^\top(kP+1) & \cdots & y^\top(kP+P-1) \end{bmatrix}^\top.$$

Then the dynamics $\tilde{y}(k) = \tilde{G}(\mathrm{e}^{j\omega})\tilde{u}(k)$ is LTI with the (l,m)-th block element:

$$\tilde{G}_{l,m}(\mathrm{e}^{j\omega}) = \sum_{s=0}^{\infty} g^l_{sP+l-m} \exp\left(-j\omega s\right),$$

where $g^t_r = C_t A_{t-1} A_{t-2} \cdots A_{t-r+1} B_{t-r}$ is the periodic impulse response.

A two-step approach

1) Lift the LTP system and estimate $\tilde{G}(\mathrm{e}^{j\omega})$ of the lifted system
   – Not a trivial problem as it is MIMO and cannot be excited separately
2) Identify the LTP system from the $\tilde{G}(\mathrm{e}^{j\omega})$ estimate
   – Extension of frequency-domain subspace methods for LTI systems

Assumptions:

- J input-output sequences of length NP with periodic inputs, $J \geq Pn_u$
- Zero mean i.i.d noise across experiments, uncorrelated with inputs
- Fast-decaying covariances inside each experiment
  $\sum_{\tau=1}^{\infty} |\tau \cdot \mathbb{E}\left[w(t)w(t-\tau)\right]| = c < \infty$

Generalized empirical transfer function estimate

$$\hat{\tilde{G}}(\mathrm{e}^{j\omega_k}) = \tilde{Y}(\mathrm{e}^{j\omega_k})\tilde{U}^{\dagger}(\mathrm{e}^{j\omega_k}),$$

- Columns of $\tilde{Y}(\mathrm{e}^{j\omega_k})$ and $\tilde{U}(\mathrm{e}^{j\omega_k})$ are DFT of each input-output sequence
- Multiple periodic trajectories are needed to satisfy the PE requirement

Properties of the generalized ETFE

- $\mathbb{E}\left[\hat{\tilde{G}}(\mathrm{e}^{j\omega_k})\right] = \tilde{G}(\mathrm{e}^{j\omega_k}),$
- Cov $\left[\hat{\tilde{G}}\right] = (\Phi_w + \rho(N))\left(\tilde{U}^{\dagger}\right)^{\mathsf{H}}\tilde{U}^{\dagger}$, Φ_w: noise spectrum, $|\rho(N)| \leq 2c/N$.
- Estimates at different frequencies are independent.

Time-aliased periodic impulse response

- $w_{l,m}(n)$: the IDFT of generalized ETFE $\hat{\hat{\tilde{G}}}_{l,m}(\mathrm{e}^{j\omega_k})$.
- According to the definition of $\tilde{G}_{l,m}(\mathrm{e}^{j\omega_k})$,

$$w_{l,m}(n) = \begin{cases} h^{l}_{nP+l-m}, & nP+l-m>0, \\ h^{l}_{(N+n)P+l-m}, & nP+l-m\leq 0, \end{cases}$$

where $h^t_r = \sum_{i=0}^{\infty} g^t_{r+iNP}$, $r = 1, 2, \cdots, NP$: time-aliased periodic impulse response.

Extension of the LTI result:

- IDFT of frequency response = time-aliased impulse response

Order-revealing decomposition for LTP systems

$$H_p^\tau = \begin{bmatrix} h_1^\tau & h_2^\tau & \cdots & h_r^\tau \\ h_2^{\tau+1} & h_3^{\tau+1} & \cdots & h_{r+1}^{\tau+1} \\ \vdots & \vdots & \ddots & \vdots \\ h_q^{\tau+P-1} & h_{q+1}^{\tau+P-1} & \cdots & h_{q+r-1}^{\tau+P-1} \end{bmatrix} = \mathcal{O}_q^\tau \left(I - \Psi_{A,\tau}^N\right)^{-1} \mathcal{C}_r^\tau,$$

where

$$\Psi_{A,t} = A_{t-1}A_{t-2}\cdots A_{t-P},$$

$$\mathcal{C}_s^\tau = \begin{bmatrix} B_{\tau-1} & A_{\tau-1}B_{\tau-2} & \cdots & A_{\tau-1}\cdots A_{\tau-s+1}B_{\tau-s} \end{bmatrix},$$

$$\mathcal{O}_s^\tau = \text{col}\left(C_\tau, C_{\tau+1}A_\tau, \cdots, C_{\tau+s-1}A_{\tau+s-2}\cdots A_\tau\right).$$

- **Key property:** rank $\left(H_p^\tau\right) = n_x$, range $\left(H_p^\tau\right) =$ range $\left(\mathcal{O}_s^\tau\right)$

The proposed algorithm

Frequency-domain subspace identification of LTP systems

1) Lift the input-output data.
2) Estimate $\tilde{G}(e^{j\omega_k})$ of the lifted system by generalized ETFE.
3) Apply IDFT on $\hat{\tilde{G}}(e^{j\omega_k})$ and find $\hat{h}_r^t$ by rearranging elements.
4) Construct H_p^τ from $\hat{h}_r^t$.
5) Find an n_x-th order approximation of range $\left(H_p^\tau\right)$ as the estimate of $\mathcal{O}_s^\tau$.
6) Estimate A_τ, C_τ from $\hat{\mathcal{O}}_s^\tau$.
7) Estimate B_τ by fitting $\hat{h}_r^t$.

Consistency property

Theorem

Let $\hat{A}_t$, $\hat{B}_t$, and $\hat{C}_t$ be the estimated state matrices by the proposed algorithm. Given the stated assumptions, there exist nonsingular periodic matrices $T_t \in \mathbb{R}^{n_x \times n_x}$, $T_t = T_{t+P}$ such that w.p. 1,

$$\lim_{N\to\infty} \left\| \begin{bmatrix} A_t & B_t \\ C_t & \mathbf{0} \end{bmatrix} - \begin{bmatrix} T_{t+1} & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} \begin{bmatrix} \hat{A}_t & \hat{B}_t \\ \hat{C}_t & \mathbf{0} \end{bmatrix} \begin{bmatrix} T_t^{-1} & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} \right\|_F = 0,$$

for a fixed choice of q, r.

Numerical example

- Compare with existing time-domain subspace algorithms:
  - MOESP algorithm in (Verhaegen and Yu 1995)
  - intersection algorithm in (Hench 1995)
  - CCA algorithm in (Cox 2018)

- Periodic input of i.i.d. unit Gaussian entries
- i.i.d. unit Gaussian noise
- $N = 50$, $J = 10 \cdot P$, n_x is known

Example 1

$$\left[\begin{array}{c|c} A_0 & B_0 \\ \hline C_0 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 0 & 0.0734 & -0.07221 \\ -6.5229 & -0.4997 & -9.6277 \\ \hline 1 & 0 & 0 \end{array}\right], \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -0.0021 & 0 & 0 \\ -0.0138 & 0.5196 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$$

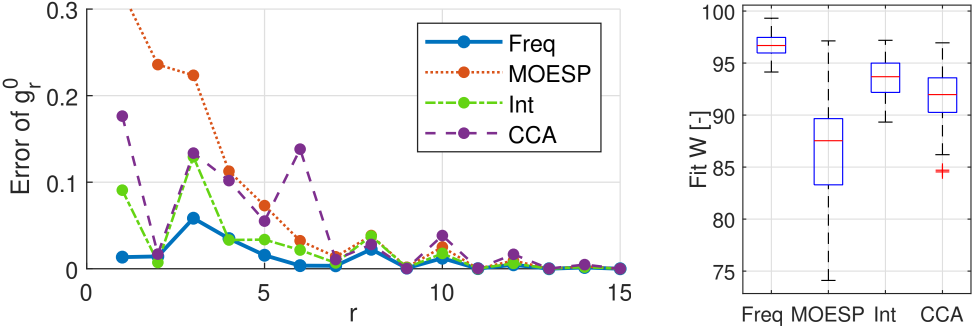

Figure: Accuracy of state-space model estimation in terms of impulse response.

Consistency of the estimate

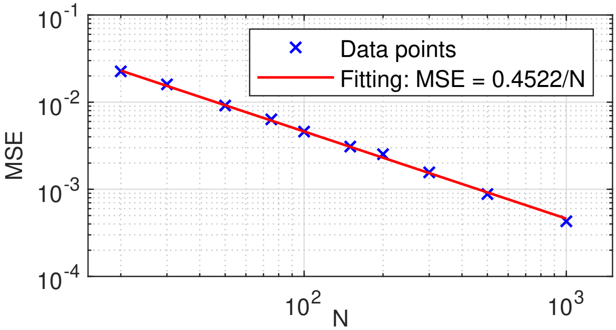

Figure: The estimate is consistent with a convergence rate of $1/N$.

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A novel subspace identification method for LTP system

- Generalized ETFE for lifted LTP system
- Order revealing decomposition of lifted LTP system

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Mingzhou Yin
myin@ethz.ch