Linear Time-Periodic System Identification with Grouped Atomic Norm Regularization

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Outline

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   - Grouped atomic norm regularization
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Periodic systems, but why?

- Periodicity comes naturally in practical systems
  - Rotating dynamics (e.g., wind turbine, Allen, Sracic, et al. 2011)
  - Periodic scheduling parameters (e.g., Felici, Wingerden, and Verhaegen 2007)
  - Periodic operating trajectory (e.g., Allen and Sracic 2009)
- Linear time-periodic (LTP) systems as an intermediate step for
  - LTV systems
  - LPV systems
  - nonlinear systems along limit cycles

The problem

LTP identification problem

Consider discrete-time stable SISO minimal LTP system

$$\begin{cases} x(t+1) &= A_t x(t) + B_t u(t) \\ y(t) &= C_t x(t) \end{cases}, A_t = A_{t+P}, B_t = B_{t+P}, C_t = C_{t+P}, \qquad (1)$$

P is the known period.

Problem: Estimate a low-order model of (1) from input sequence $u(t)$ and noisy output $z(t) = y(t) + w(t), w(t) \sim \mathcal{N}(0, \sigma^2)$.

On the shoulders of LTI systems

- LTP systems as structured LTI systems

Lifting

Concatenate inputs and outputs of one period

$$\tilde{u}_\tau(k) = \begin{bmatrix} u^\top(kP+\tau) & u^\top(kP+\tau+1) & \cdots & u^\top(kP+\tau+P-1) \end{bmatrix}^\top,$$

$$\tilde{y}_\tau(k) = \begin{bmatrix} y^\top(kP+\tau) & y^\top(kP+\tau+1) & \cdots & y^\top(kP+\tau+P-1) \end{bmatrix}^\top,$$

where $\tau = 0, 1, \cdots, P-1$. Then the dynamics $\tilde{y}_\tau(k) = \tilde{G}_\tau(q^P)\tilde{u}_\tau(k)$ is LTI.

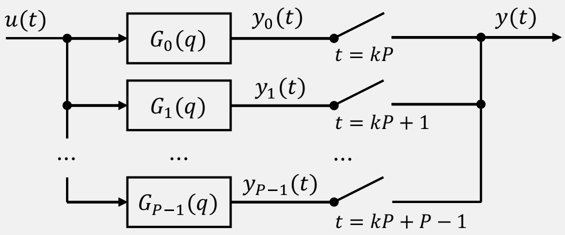
- Closely connected to subspace algorithms (Verhaegen and Yu 1995; Hench 1995)

On the shoulders of LTI systems

- LTP systems as structured LTI systems

Switching

LTP systems can be modeled as switching between P LTI systems $\{G_\tau(q)\}_{\tau=0}^{P-1}$.



What's missing

LTP systems as **structured** LTI systems (Bittanti and Colaneri 2000).

What's the structure?

- **Causality:** Lifted model $\tilde{G}_\tau(q^P)$ has internal causality constraints.
- **Uniformity:** Model order for $\tilde{G}_\tau(q^P)$ and $G_\tau(q)$ should be the same for all τ.
- **Realizability:** Separately identified $\tilde{G}_\tau(q^P)$ and $G_\tau(q)$ cannot be realized to the LTP model.

Separating models and structure

Regularization method

$$\underset{\theta}{\text{minimize}}\ V\left(\theta, \begin{bmatrix}\mathbf{u}\\ \mathbf{z}\end{bmatrix}\right) + \gamma \cdot J(\theta),$$

where θ denotes the parameters to be estimated.

$V(\cdot,\cdot)$ How data adheres to the model

$J(\cdot)$ How the model adheres to structural requirements

- Possible to use simple models (e.g., FIR models) for complex structure
- Typical structural requirements: stability, continuity, & low complexity
- **In this work:** to include uniformity and realizability

A naïve idea...

- Consider FIR models of switching models $G_\tau(q)$

$$y(kP+\tau) = \sum_{i=1}^{N} g_i^\tau u(kP+\tau-i)$$

- Forgetting all structural requirements, minimize model fitting errors

$$\underset{\mathbf{g}}{\text{minimize}} \sum_{\tau=0}^{P-1} \sum_{k=0}^{n-1} \left[z(kP+\tau) - \sum_{i=1}^{N} g_i^\tau u(kP+\tau-i) \right]^2 \qquad \text{(LS)}$$

Low-complexity regularizers in LTI

- Enforce a low order structure on $G_\tau(q)$

Hankel nuclear norm regularization

$$\text{rank}\left(\mathcal{H}(\mathbf{g}^\tau)\right) = \text{rank}\left(\begin{bmatrix} g_1^\tau & g_2^\tau & \cdots & g_{N-m+1}^\tau \\ g_2^\tau & g_3^\tau & \cdots & g_{N-m+2}^\tau \\ \vdots & \vdots & \ddots & \vdots \\ g_m^\tau & g_{m+1}^\tau & \cdots & g_N^\tau \end{bmatrix}\right) = n_x.$$

Convex surrogate of the rank function: nuclear norm $\|\mathcal{H}(\mathbf{g}^\tau)\|_*$.

$$\underset{\mathbf{g}}{\text{minimize}}\ V_{\text{LS}}\left(\mathbf{g}, \begin{bmatrix}\mathbf{u}\\ \mathbf{z}\end{bmatrix}\right) + \sum_{\tau=1}^{P} \beta_\tau \|\mathcal{H}(\mathbf{g}^\tau)\|_* \qquad \text{(Hank)}$$

Low-complexity regularizers in LTI

Atomic norm regularization (Shah et al. 2012)

The switching models $G_\tau(q)$ can be decomposed as a linear combination of stable first order models

$$G_\tau(q) = \sum_{w \in \mathbb{D}} c_w^\tau \cdot a_w(q) \approx \sum_{k=1}^{n_p} c_k^\tau \cdot a_{w_k}(q) := \mathbf{c}_\tau^T \mathbf{a}(q), \quad a_w(q) = \frac{1-|w|^2}{q-w}$$

Then,

$$\mathrm{card}(\mathbf{c}_\tau) = n_x.$$

Convex surrogate of the cardinality function: l_1 norm $\|\mathbf{c}_\tau\|_1$, also known as the atomic norm of $G_\tau(q)$.

Low-complexity regularizers in LTI

$$\underset{\mathbf{c}}{\text{minimize}}\ V_{\text{LS}}\left(\mathbf{g}^a\mathbf{c}, \begin{bmatrix}\mathbf{u}\\ \mathbf{z}\end{bmatrix}\right) + \sum_{\tau=1}^{P} \beta_\tau \left\|\mathbf{c}_\tau\right\|_1, \tag{Atom}$$

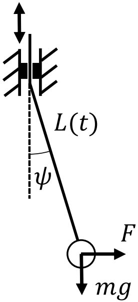
where $\mathbf{g}^a$ is the truncated impulse responses of $\mathbf{a}(q)$.

Compared with (Hank),

Pros: Stability guarantee, better scalability

Cons: Pole location approximation, impulse response truncation

A variable-length pendulum example



Model as a discrete-time SISO LTP system from F to ψ with $P = 4$.

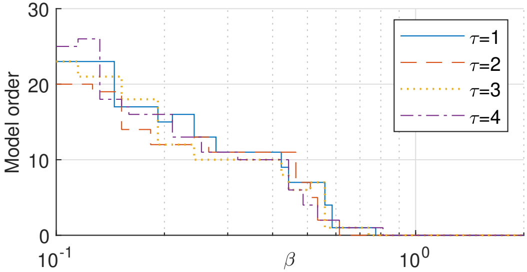
Can we get a switching LTP model with uniform order by fine tuning of β_τ?

$$\ddot{\psi} = -\frac{g}{L(t)} \sin\psi + \frac{2\omega l \sin\omega t}{L(t)} \dot{\psi} + \frac{1}{mL(t)} F \cos\psi,$$

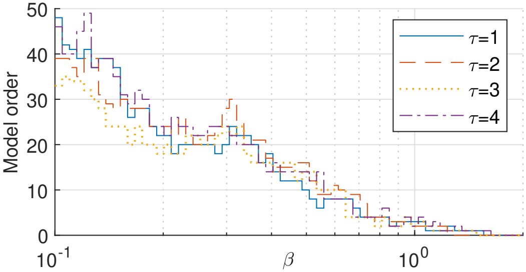
$$L(t) = L_0 + l \cos\omega t$$

A pitfall...

... *No.* There is barely any consensus on the model order.



Hankel nuclear norm



Atomic norm

A key observation to satisfy LTP requirements

$$G_\tau(q) = C(\tau)(q^P \mathbb{I} - \Psi_{A,\tau})^{-1}\mathcal{B}_\tau(q),$$

where

$$\Psi_{A,\tau} = A(\tau-1)A(\tau-2)\cdots A(\tau-P)$$

$$\mathcal{B}_\tau(q) = \sum_{i=0}^{P-1} A(\tau-1)A(\tau-2)\cdots A(\tau+i-P+1)B(\tau+i)\cdot q^i.$$

- The poles of $G_\tau(q)$ are P-th root of the poles of $\Psi_{A,\tau}$, which happens to be independent of τ (Bittanti 1986).
- **The active set of $\mathbf{c}_\tau$ is the same for all τ.**

Connecting switching models

Proposed method: Grouped atomic norm regularization

Let $\mathbf{c} = [\mathbf{c}_0 \; \mathbf{c}_1 \; \cdots \; \mathbf{c}_{P-1}]$, $\mathbf{c}$ is sparse for each column, but non-sparse or all zero for each row.

Then,

$$\text{card}\left(\left\|\mathbf{c}^{(k)}\right\|_2\right) = n_x, \quad \mathbf{c}^{(k)}\text{: } k\text{-th row of } \mathbf{c}$$

Convex surrogate: sum-of-norms $\sum_{k=1}^{n_p} \left\|\mathbf{c}^{(k)}\right\|_2$.

$\mathbf{c} =$

Grouped atomic norm regularization

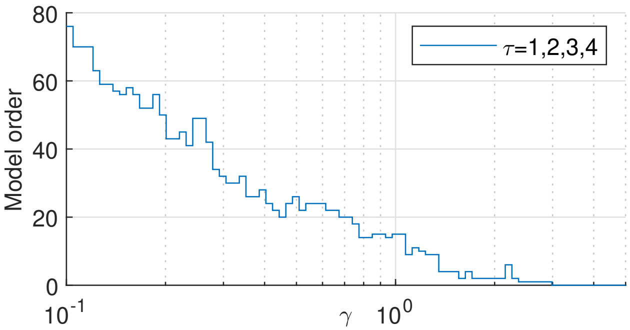
$$\underset{\mathbf{c}}{\text{minimize}}\; V_{\mathsf{LS}}\left(\mathbf{g}^a\mathbf{c}, \begin{bmatrix}\mathbf{u}\\ \mathbf{z}\end{bmatrix}\right) + \gamma \sum_{k=1}^{n_p} \left\|\mathbf{c}^{(k)}\right\|_2 \qquad \text{(GAtom)}$$

Characteristics of this estimator:

- Uniformity is guaranteed
- Realizability is promising *(ongoing work)*
- Low-complexity and stability maintained as in (Atom)

Check with the pendulum example

It's working!

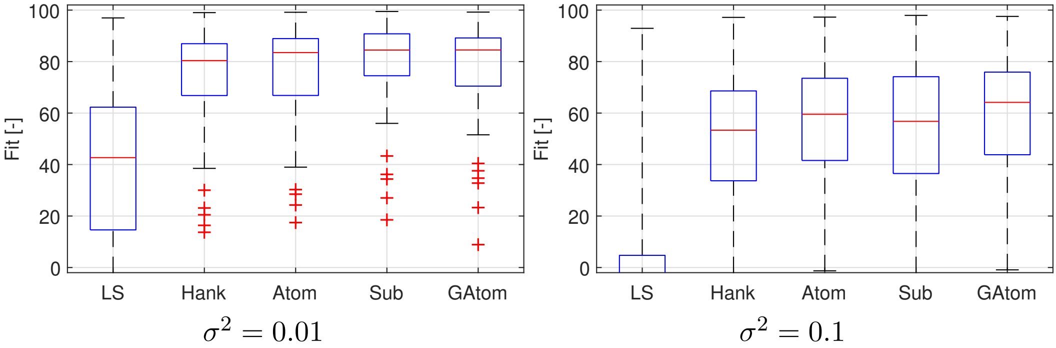


Grouped atomic norm

Model fitting on random systems

- 100 randomly generated LTP systems
- $P = 2$, system order between 2 and 10
- 500 data points
- Unit Gaussian inputs
- Two noise levels: $\sigma^2 = 0.1, 0.01$
- Hyperparameter tuning by cross-validation
- Fitting metric: $W = 100 \cdot \left(1 - \frac{\|\mathbf{g}-\mathbf{g}_0\|_2}{\|\mathbf{g}_0-\bar{g}\|_2}\right)$

Results



	$\sigma^2 = 0.01$					$\sigma^2 = 0.1$				
	LS	**Hank**	**Atom**	**Sub**	**GAtom**	**LS**	**Hank**	**Atom**	**Sub**	**GAtom**
Median	42.7	80.4	83.5	84.5	84.5	-79.5	53.3	59.5	56.8	64.2
Std	86.2	37.8	38.5	22.6	39.9	203.5	36.5	36.2	48.3	34.1

Conclusion & outlook

- LTP system identification cannot be fully tackled by LTI reformulation
- Periodic structure is the key to estimating LTP systems
- Such structure can be encoded by regularization methods

Further problems:

- Realization of the grouped atomic norm estimator
- Active selection of first-order basis model